



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

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Contents

I	Basics	1
1	Introduction to Book	3
1.1	The Language of Mathematics	3
II	Grade 10	5
2	Review of Past Work	7
2.1	Introduction	7
2.2	What is a number?	7
2.3	Sets	7
2.4	Letters and Arithmetic	8
2.5	Addition and Subtraction	9
2.6	Multiplication and Division	9
2.7	Brackets	9
2.8	Negative Numbers	10
2.8.1	What is a negative number?	10
2.8.2	Working with Negative Numbers	11
2.8.3	Living Without the Number Line	12
2.9	Rearranging Equations	13
2.10	Fractions and Decimal Numbers	15
2.11	Scientific Notation	16
2.12	Real Numbers	16
2.12.1	Natural Numbers	17
2.12.2	Integers	17
2.12.3	Rational Numbers	17
2.12.4	Irrational Numbers	19
2.13	Mathematical Symbols	20
2.14	Infinity	20
2.15	End of Chapter Exercises	21
3	Rational Numbers - Grade 10	23
3.1	Introduction	23
3.2	The Big Picture of Numbers	23
3.3	Definition	23

3.4	Forms of Rational Numbers	24
3.5	Converting Terminating Decimals into Rational Numbers	25
3.6	Converting Repeating Decimals into Rational Numbers	25
3.7	Summary	26
3.8	End of Chapter Exercises	27
4	Exponentials - Grade 10	29
4.1	Introduction	29
4.2	Definition	29
4.3	Laws of Exponents	30
4.3.1	Exponential Law 1: $a^0 = 1$	30
4.3.2	Exponential Law 2: $a^m \times a^n = a^{m+n}$	30
4.3.3	Exponential Law 3: $a^{-n} = \frac{1}{a^n}, a \neq 0$	31
4.3.4	Exponential Law 4: $a^m \div a^n = a^{m-n}$	32
4.3.5	Exponential Law 5: $(ab)^n = a^n b^n$	32
4.3.6	Exponential Law 6: $(a^m)^n = a^{mn}$	33
4.4	End of Chapter Exercises	34
5	Estimating Surds - Grade 10	37
5.1	Introduction	37
5.2	Drawing Surds on the Number Line (Optional)	38
5.3	End of Chapter Exercises	39
6	Irrational Numbers and Rounding Off - Grade 10	41
6.1	Introduction	41
6.2	Irrational Numbers	41
6.3	Rounding Off	42
6.4	End of Chapter Exercises	43
7	Number Patterns - Grade 10	45
7.1	Common Number Patterns	45
7.1.1	Special Sequences	46
7.2	Make your own Number Patterns	46
7.3	Notation	47
7.3.1	Patterns and Conjecture	49
7.4	Exercises	50
8	Finance - Grade 10	53
8.1	Introduction	53
8.2	Foreign Exchange Rates	53
8.2.1	How much is R1 really worth?	53
8.2.2	Cross Currency Exchange Rates	56
8.2.3	Enrichment: Fluctuating exchange rates	57
8.3	Being Interested in Interest	58

8.4	Simple Interest	59
8.4.1	Other Applications of the Simple Interest Formula	61
8.5	Compound Interest	63
8.5.1	Fractions add up to the Whole	65
8.5.2	The Power of Compound Interest	65
8.5.3	Other Applications of Compound Growth	67
8.6	Summary	68
8.6.1	Definitions	68
8.6.2	Equations	68
8.7	End of Chapter Exercises	69
9	Products and Factors - Grade 10	71
9.1	Introduction	71
9.2	Recap of Earlier Work	71
9.2.1	Parts of an Expression	71
9.2.2	Product of Two Binomials	71
9.2.3	Factorisation	72
9.3	More Products	74
9.4	Factorising a Quadratic	76
9.5	Factorisation by Grouping	79
9.6	Simplification of Fractions	80
9.7	End of Chapter Exercises	82
10	Equations and Inequalities - Grade 10	83
10.1	Strategy for Solving Equations	83
10.2	Solving Linear Equations	84
10.3	Solving Quadratic Equations	89
10.4	Exponential Equations of the form $ka^{(x+p)} = m$	93
10.4.1	Algebraic Solution	93
10.5	Linear Inequalities	96
10.6	Linear Simultaneous Equations	99
10.6.1	Finding solutions	99
10.6.2	Graphical Solution	99
10.6.3	Solution by Substitution	101
10.7	Mathematical Models	103
10.7.1	Introduction	103
10.7.2	Problem Solving Strategy	104
10.7.3	Application of Mathematical Modelling	104
10.7.4	End of Chapter Exercises	106
10.8	Introduction to Functions and Graphs	107
10.9	Functions and Graphs in the Real-World	107
10.10	Recap	107

10.10.1 Variables and Constants	107
10.10.2 Relations and Functions	108
10.10.3 The Cartesian Plane	108
10.10.4 Drawing Graphs	109
10.10.5 Notation used for Functions	110
10.11 Characteristics of Functions - All Grades	112
10.11.1 Dependent and Independent Variables	112
10.11.2 Domain and Range	113
10.11.3 Intercepts with the Axes	113
10.11.4 Turning Points	114
10.11.5 Asymptotes	114
10.11.6 Lines of Symmetry	114
10.11.7 Intervals on which the Function Increases/Decreases	114
10.11.8 Discrete or Continuous Nature of the Graph	114
10.12 Graphs of Functions	116
10.12.1 Functions of the form $y = ax + q$	116
10.12.2 Functions of the Form $y = ax^2 + q$	120
10.12.3 Functions of the Form $y = \frac{a}{x} + q$	125
10.12.4 Functions of the Form $y = ab^{(x)} + q$	129
10.13 End of Chapter Exercises	133
11 Average Gradient - Grade 10 Extension	135
11.1 Introduction	135
11.2 Straight-Line Functions	135
11.3 Parabolic Functions	136
11.4 End of Chapter Exercises	138
12 Geometry Basics	139
12.1 Introduction	139
12.2 Points and Lines	139
12.3 Angles	140
12.3.1 Measuring angles	141
12.3.2 Special Angles	141
12.3.3 Special Angle Pairs	143
12.3.4 Parallel Lines intersected by Transversal Lines	143
12.4 Polygons	147
12.4.1 Triangles	147
12.4.2 Quadrilaterals	152
12.4.3 Other polygons	155
12.4.4 Extra	156
12.5 Exercises	157
12.5.1 Challenge Problem	159

13 Geometry - Grade 10	161
13.1 Introduction	161
13.2 Right Prisms and Cylinders	161
13.2.1 Surface Area	162
13.2.2 Volume	164
13.3 Polygons	167
13.3.1 Similarity of Polygons	167
13.4 Co-ordinate Geometry	171
13.4.1 Introduction	171
13.4.2 Distance between Two Points	172
13.4.3 Calculation of the Gradient of a Line	173
13.4.4 Midpoint of a Line	174
13.5 Transformations	177
13.5.1 Translation of a Point	177
13.5.2 Reflection of a Point	179
13.6 End of Chapter Exercises	185
14 Trigonometry - Grade 10	189
14.1 Introduction	189
14.2 Where Trigonometry is Used	190
14.3 Similarity of Triangles	190
14.4 Definition of the Trigonometric Functions	191
14.5 Simple Applications of Trigonometric Functions	195
14.5.1 Height and Depth	195
14.5.2 Maps and Plans	197
14.6 Graphs of Trigonometric Functions	199
14.6.1 Graph of $\sin \theta$	199
14.6.2 Functions of the form $y = a \sin(x) + q$	200
14.6.3 Graph of $\cos \theta$	202
14.6.4 Functions of the form $y = a \cos(x) + q$	202
14.6.5 Comparison of Graphs of $\sin \theta$ and $\cos \theta$	204
14.6.6 Graph of $\tan \theta$	204
14.6.7 Functions of the form $y = a \tan(x) + q$	205
14.7 End of Chapter Exercises	208
15 Statistics - Grade 10	211
15.1 Introduction	211
15.2 Recap of Earlier Work	211
15.2.1 Data and Data Collection	211
15.2.2 Methods of Data Collection	212
15.2.3 Samples and Populations	213
15.3 Example Data Sets	213

15.3.1 Data Set 1: Tossing a Coin	213
15.3.2 Data Set 2: Casting a die	213
15.3.3 Data Set 3: Mass of a Loaf of Bread	214
15.3.4 Data Set 4: Global Temperature	214
15.3.5 Data Set 5: Price of Petrol	215
15.4 Grouping Data	215
15.4.1 Exercises - Grouping Data	216
15.5 Graphical Representation of Data	217
15.5.1 Bar and Compound Bar Graphs	217
15.5.2 Histograms and Frequency Polygons	217
15.5.3 Pie Charts	219
15.5.4 Line and Broken Line Graphs	220
15.5.5 Exercises - Graphical Representation of Data	221
15.6 Summarising Data	222
15.6.1 Measures of Central Tendency	222
15.6.2 Measures of Dispersion	225
15.6.3 Exercises - Summarising Data	228
15.7 Misuse of Statistics	229
15.7.1 Exercises - Misuse of Statistics	230
15.8 Summary of Definitions	232
15.9 Exercises	232
16 Probability - Grade 10	235
16.1 Introduction	235
16.2 Random Experiments	235
16.2.1 Sample Space of a Random Experiment	235
16.3 Probability Models	238
16.3.1 Classical Theory of Probability	239
16.4 Relative Frequency vs. Probability	240
16.5 Project Idea	242
16.6 Probability Identities	242
16.7 Mutually Exclusive Events	243
16.8 Complementary Events	244
16.9 End of Chapter Exercises	246
III Grade 11	249
17 Exponents - Grade 11	251
17.1 Introduction	251
17.2 Laws of Exponents	251
17.2.1 Exponential Law 7: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$	251
17.3 Exponentials in the Real-World	253
17.4 End of chapter Exercises	254

18 Surds - Grade 11	255
18.1 Surd Calculations	255
18.1.1 Surd Law 1: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$	255
18.1.2 Surd Law 2: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	255
18.1.3 Surd Law 3: $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	256
18.1.4 Like and Unlike Surds	256
18.1.5 Simplest Surd form	257
18.1.6 Rationalising Denominators	258
18.2 End of Chapter Exercises	259
19 Error Margins - Grade 11	261
20 Quadratic Sequences - Grade 11	265
20.1 Introduction	265
20.2 What is a <i>quadratic sequence</i> ?	265
20.3 End of chapter Exercises	269
21 Finance - Grade 11	271
21.1 Introduction	271
21.2 Depreciation	271
21.3 Simple Depreciation (it really is simple!)	271
21.4 Compound Depreciation	274
21.5 Present Values or Future Values of an Investment or Loan	276
21.5.1 Now or Later	276
21.6 Finding i	278
21.7 Finding n - Trial and Error	279
21.8 Nominal and Effective Interest Rates	280
21.8.1 The General Formula	281
21.8.2 De-coding the Terminology	282
21.9 Formulae Sheet	284
21.9.1 Definitions	284
21.9.2 Equations	285
21.10 End of Chapter Exercises	285
22 Solving Quadratic Equations - Grade 11	287
22.1 Introduction	287
22.2 Solution by Factorisation	287
22.3 Solution by Completing the Square	290
22.4 Solution by the Quadratic Formula	293
22.5 Finding an equation when you know its roots	296
22.6 End of Chapter Exercises	299

23 Solving Quadratic Inequalities - Grade 11	301
23.1 Introduction	301
23.2 Quadratic Inequalities	301
23.3 End of Chapter Exercises	304
24 Solving Simultaneous Equations - Grade 11	307
24.1 Graphical Solution	307
24.2 Algebraic Solution	309
25 Mathematical Models - Grade 11	313
25.1 Real-World Applications: Mathematical Models	313
25.2 End of Chapter Exercises	317
26 Quadratic Functions and Graphs - Grade 11	321
26.1 Introduction	321
26.2 Functions of the Form $y = a(x + p)^2 + q$	321
26.2.1 Domain and Range	322
26.2.2 Intercepts	323
26.2.3 Turning Points	324
26.2.4 Axes of Symmetry	325
26.2.5 Sketching Graphs of the Form $f(x) = a(x + p)^2 + q$	325
26.2.6 Writing an equation of a shifted parabola	327
26.3 End of Chapter Exercises	327
27 Hyperbolic Functions and Graphs - Grade 11	329
27.1 Introduction	329
27.2 Functions of the Form $y = \frac{a}{x+p} + q$	329
27.2.1 Domain and Range	330
27.2.2 Intercepts	331
27.2.3 Asymptotes	332
27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$	333
27.3 End of Chapter Exercises	333
28 Exponential Functions and Graphs - Grade 11	335
28.1 Introduction	335
28.2 Functions of the Form $y = ab^{(x+p)} + q$	335
28.2.1 Domain and Range	336
28.2.2 Intercepts	337
28.2.3 Asymptotes	338
28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$	338
28.3 End of Chapter Exercises	339
29 Gradient at a Point - Grade 11	341
29.1 Introduction	341
29.2 Average Gradient	341
29.3 End of Chapter Exercises	344

30 Linear Programming - Grade 11	345
30.1 Introduction	345
30.2 Terminology	345
30.2.1 Decision Variables	345
30.2.2 Objective Function	345
30.2.3 Constraints	346
30.2.4 Feasible Region and Points	346
30.2.5 The Solution	346
30.3 Example of a Problem	347
30.4 Method of Linear Programming	347
30.5 Skills you will need	347
30.5.1 Writing Constraint Equations	347
30.5.2 Writing the Objective Function	348
30.5.3 Solving the Problem	350
30.6 End of Chapter Exercises	352
31 Geometry - Grade 11	357
31.1 Introduction	357
31.2 Right Pyramids, Right Cones and Spheres	357
31.3 Similarity of Polygons	360
31.4 Triangle Geometry	361
31.4.1 Proportion	361
31.5 Co-ordinate Geometry	368
31.5.1 Equation of a Line between Two Points	368
31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line	371
31.5.3 Inclination of a Line	371
31.6 Transformations	373
31.6.1 Rotation of a Point	373
31.6.2 Enlargement of a Polygon 1	376
32 Trigonometry - Grade 11	381
32.1 History of Trigonometry	381
32.2 Graphs of Trigonometric Functions	381
32.2.1 Functions of the form $y = \sin(k\theta)$	381
32.2.2 Functions of the form $y = \cos(k\theta)$	383
32.2.3 Functions of the form $y = \tan(k\theta)$	384
32.2.4 Functions of the form $y = \sin(\theta + p)$	385
32.2.5 Functions of the form $y = \cos(\theta + p)$	386
32.2.6 Functions of the form $y = \tan(\theta + p)$	387
32.3 Trigonometric Identities	389
32.3.1 Deriving Values of Trigonometric Functions for 30° , 45° and 60°	389
32.3.2 Alternate Definition for $\tan \theta$	391

32.3.3	A Trigonometric Identity	392
32.3.4	Reduction Formula	394
32.4	Solving Trigonometric Equations	399
32.4.1	Graphical Solution	399
32.4.2	Algebraic Solution	401
32.4.3	Solution using CAST diagrams	403
32.4.4	General Solution Using Periodicity	405
32.4.5	Linear Trigonometric Equations	406
32.4.6	Quadratic and Higher Order Trigonometric Equations	406
32.4.7	More Complex Trigonometric Equations	407
32.5	Sine and Cosine Identities	409
32.5.1	The Sine Rule	409
32.5.2	The Cosine Rule	412
32.5.3	The Area Rule	414
32.6	Exercises	416
33	Statistics - Grade 11	419
33.1	Introduction	419
33.2	Standard Deviation and Variance	419
33.2.1	Variance	419
33.2.2	Standard Deviation	421
33.2.3	Interpretation and Application	423
33.2.4	Relationship between Standard Deviation and the Mean	424
33.3	Graphical Representation of Measures of Central Tendency and Dispersion	424
33.3.1	Five Number Summary	424
33.3.2	Box and Whisker Diagrams	425
33.3.3	Cumulative Histograms	426
33.4	Distribution of Data	428
33.4.1	Symmetric and Skewed Data	428
33.4.2	Relationship of the Mean, Median, and Mode	428
33.5	Scatter Plots	429
33.6	Misuse of Statistics	432
33.7	End of Chapter Exercises	435
34	Independent and Dependent Events - Grade 11	437
34.1	Introduction	437
34.2	Definitions	437
34.2.1	Identification of Independent and Dependent Events	438
34.3	End of Chapter Exercises	441
IV	Grade 12	443
35	Logarithms - Grade 12	445
35.1	Definition of Logarithms	445

35.2	Logarithm Bases	446
35.3	Laws of Logarithms	447
35.4	Logarithm Law 1: $\log_a 1 = 0$	447
35.5	Logarithm Law 2: $\log_a(a) = 1$	448
35.6	Logarithm Law 3: $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$	448
35.7	Logarithm Law 4: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	449
35.8	Logarithm Law 5: $\log_a(x^b) = b \log_a(x)$	450
35.9	Logarithm Law 6: $\log_a(\sqrt[b]{x}) = \frac{\log_a(x)}{b}$	450
35.10	Solving simple log equations	452
35.10.1	Exercises	454
35.11	Logarithmic applications in the Real World	454
35.11.1	Exercises	455
35.12	End of Chapter Exercises	455
36	Sequences and Series - Grade 12	457
36.1	Introduction	457
36.2	Arithmetic Sequences	457
36.2.1	General Equation for the n^{th} -term of an Arithmetic Sequence	458
36.3	Geometric Sequences	459
36.3.1	Example - A Flu Epidemic	459
36.3.2	General Equation for the n^{th} -term of a Geometric Sequence	461
36.3.3	Exercises	461
36.4	Recursive Formulae for Sequences	462
36.5	Series	463
36.5.1	Some Basics	463
36.5.2	Sigma Notation	463
36.6	Finite Arithmetic Series	465
36.6.1	General Formula for a Finite Arithmetic Series	466
36.6.2	Exercises	467
36.7	Finite Squared Series	468
36.8	Finite Geometric Series	469
36.8.1	Exercises	470
36.9	Infinite Series	471
36.9.1	Infinite Geometric Series	471
36.9.2	Exercises	472
36.10	End of Chapter Exercises	472
37	Finance - Grade 12	477
37.1	Introduction	477
37.2	Finding the Length of the Investment or Loan	477
37.3	A Series of Payments	478
37.3.1	Sequences and Series	479

37.3.2 Present Values of a series of Payments	479
37.3.3 Future Value of a series of Payments	484
37.3.4 Exercises - Present and Future Values	485
37.4 Investments and Loans	485
37.4.1 Loan Schedules	485
37.4.2 Exercises - Investments and Loans	489
37.4.3 Calculating Capital Outstanding	489
37.5 Formulae Sheet	489
37.5.1 Definitions	490
37.5.2 Equations	490
37.6 End of Chapter Exercises	490
38 Factorising Cubic Polynomials - Grade 12	493
38.1 Introduction	493
38.2 The Factor Theorem	493
38.3 Factorisation of Cubic Polynomials	494
38.4 Exercises - Using Factor Theorem	496
38.5 Solving Cubic Equations	496
38.5.1 Exercises - Solving of Cubic Equations	498
38.6 End of Chapter Exercises	498
39 Functions and Graphs - Grade 12	501
39.1 Introduction	501
39.2 Definition of a Function	501
39.2.1 Exercises	501
39.3 Notation used for Functions	502
39.4 Graphs of Inverse Functions	502
39.4.1 Inverse Function of $y = ax + q$	503
39.4.2 Exercises	504
39.4.3 Inverse Function of $y = ax^2$	504
39.4.4 Exercises	504
39.4.5 Inverse Function of $y = a^x$	506
39.4.6 Exercises	506
39.5 End of Chapter Exercises	507
40 Differential Calculus - Grade 12	509
40.1 Why do I have to learn this stuff?	509
40.2 Limits	510
40.2.1 A Tale of Achilles and the Tortoise	510
40.2.2 Sequences, Series and Functions	511
40.2.3 Limits	512
40.2.4 Average Gradient and Gradient at a Point	516
40.3 Differentiation from First Principles	519

40.4 Rules of Differentiation	521
40.4.1 Summary of Differentiation Rules	522
40.5 Applying Differentiation to Draw Graphs	523
40.5.1 Finding Equations of Tangents to Curves	523
40.5.2 Curve Sketching	524
40.5.3 Local minimum, Local maximum and Point of Inflexion	529
40.6 Using Differential Calculus to Solve Problems	530
40.6.1 Rate of Change problems	534
40.7 End of Chapter Exercises	535
41 Linear Programming - Grade 12	539
41.1 Introduction	539
41.2 Terminology	539
41.2.1 Feasible Region and Points	539
41.3 Linear Programming and the Feasible Region	540
41.4 End of Chapter Exercises	546
42 Geometry - Grade 12	549
42.1 Introduction	549
42.2 Circle Geometry	549
42.2.1 Terminology	549
42.2.2 Axioms	550
42.2.3 Theorems of the Geometry of Circles	550
42.3 Co-ordinate Geometry	566
42.3.1 Equation of a Circle	566
42.3.2 Equation of a Tangent to a Circle at a Point on the Circle	569
42.4 Transformations	571
42.4.1 Rotation of a Point about an angle θ	571
42.4.2 Characteristics of Transformations	573
42.4.3 Characteristics of Transformations	573
42.5 Exercises	574
43 Trigonometry - Grade 12	577
43.1 Compound Angle Identities	577
43.1.1 Derivation of $\sin(\alpha + \beta)$	577
43.1.2 Derivation of $\sin(\alpha - \beta)$	578
43.1.3 Derivation of $\cos(\alpha + \beta)$	578
43.1.4 Derivation of $\cos(\alpha - \beta)$	579
43.1.5 Derivation of $\sin 2\alpha$	579
43.1.6 Derivation of $\cos 2\alpha$	579
43.1.7 Problem-solving Strategy for Identities	580
43.2 Applications of Trigonometric Functions	582
43.2.1 Problems in Two Dimensions	582

43.2.2 Problems in 3 dimensions	584
43.3 Other Geometries	586
43.3.1 Taxicab Geometry	586
43.3.2 Manhattan distance	586
43.3.3 Spherical Geometry	587
43.3.4 Fractal Geometry	588
43.4 End of Chapter Exercises	589
44 Statistics - Grade 12	591
44.1 Introduction	591
44.2 A Normal Distribution	591
44.3 Extracting a Sample Population	593
44.4 Function Fitting and Regression Analysis	594
44.4.1 The Method of Least Squares	596
44.4.2 Using a calculator	597
44.4.3 Correlation coefficients	599
44.5 Exercises	600
45 Combinations and Permutations - Grade 12	603
45.1 Introduction	603
45.2 Counting	603
45.2.1 Making a List	603
45.2.2 Tree Diagrams	604
45.3 Notation	604
45.3.1 The Factorial Notation	604
45.4 The Fundamental Counting Principle	604
45.5 Combinations	605
45.5.1 Counting Combinations	605
45.5.2 Combinatorics and Probability	606
45.6 Permutations	606
45.6.1 Counting Permutations	607
45.7 Applications	608
45.8 Exercises	610
V Exercises	613
46 General Exercises	615
47 Exercises - Not covered in Syllabus	617
A GNU Free Documentation License	619

Chapter 20

Quadratic Sequences - Grade 11

20.1 Introduction

In Grade 10, you learned about arithmetic sequences, where the difference between consecutive terms was constant. In this chapter we learn about quadratic sequences.

20.2 What is a quadratic sequence?

**Definition: Quadratic Sequence**

A quadratic sequence is a sequence of numbers in which the second differences between each consecutive term differ by the same amount, called a common second difference.

For example,

$$1; 2; 4; 7; 11; \dots \quad (20.1)$$

is a quadratic sequence. Let us see why ...

If we take the difference between consecutive terms, then:

$$\begin{aligned} a_2 - a_1 &= 2 - 1 = 1 \\ a_3 - a_2 &= 4 - 2 = 2 \\ a_4 - a_3 &= 7 - 4 = 3 \\ a_5 - a_4 &= 11 - 7 = 4 \end{aligned}$$

We then work out the *second differences*, which is simply obtained by taking the difference between the consecutive differences $\{1; 2; 3; 4; \dots\}$ obtained above:

$$\begin{aligned} 2 - 1 &= 1 \\ 3 - 2 &= 1 \\ 4 - 3 &= 1 \\ &\dots \end{aligned}$$

We then see that the second differences are equal to 1. Thus, (20.1) is a *quadratic sequence*.

Note that the differences between consecutive terms (that is, the first differences) of a quadratic sequence form a sequence where there is a constant difference between consecutive terms. In the above example, the sequence of $\{1; 2; 3; 4; \dots\}$, which is formed by taking the differences between consecutive terms of (20.1), has a linear formula of the kind $ax + b$.



Exercise: Quadratic Sequences

The following are also examples of quadratic sequences:

3; 6; 10; 15; 21; ...

4; 9; 16; 25; 36; ...

7; 17; 31; 49; 71; ...

2; 10; 26; 50; 82; ...

31; 30; 27; 22; 15; ...

Can you calculate the common second difference for each of the above examples?



Worked Example 93: Quadratic sequence

Question: Write down the next two terms and find a formula for the n^{th} term of the sequence 5, 12, 23, 38, ..., ...

Answer

Step 1 : Find the first differences between the terms.

i.e. 7, 11, 15

Step 2 : Find the 2nd differences between the terms.

the second difference is 4.

So continuing the sequence, the differences between each term will be:

$$15 + 4 = 19$$

$$19 + 4 = 23$$

Step 3 : Finding the next two terms.

So the next two terms in the sequence will be:

$$38 + 19 = 57$$

$$57 + 23 = 80$$

So the sequence will be: 5, 12, 23, 38, 57, 80

Step 4 : We now need to find the formula for this sequence.

We know that the first difference is 4. The start of the formula will therefore be $2n^2$.

Step 5 : We now need to work out the next part of the sequence.

If $n = 1$, you have to get the value of term 1, which is 5 in this particular sequence.

The difference between $2n^2 = 2$ and original number (5) is 3, which leads to $n + 2$.

Check if it works for the second term, i.e. when $n = 2$.

Then $2n^2 = 8$. The difference between term 2 (12) and 8 is 4, which can be written as $n + 2$.

So for the sequence 5, 12, 23, 38, ... the formula for the n^{th} term is $2n^2 + n + 2$.

General Case

If the sequence is quadratic, the n^{th} term should be $T_n = an^2 + bn + c$

TERMS	$a + b + c$	$4a + 2b + c$	$9a + 3b + c$
1 st difference	$3a + b$	$5a + b$	$7a + b$
2 nd difference	$2a$	$2a$	$2a$

In each case, the 2nd difference is $2a$. This fact can be used to find a , then b then c .



Worked Example 94: Quadratic Sequence

Question: The following sequence is quadratic: 8, 22, 42, 68, ... Find the rule.

Answer

Step 1 : Assume that the rule is $an^2 + bn + c$

TERMS	8	22	42	68
1 st difference	14	20	26	
2 nd difference		6	6	6

Step 2 : Determine values for a, b and c

$$\text{Then } 2a = 6 \text{ which gives } a = 3$$

$$\text{And } 3a + b = 14 \rightarrow 9 + b = 14 \rightarrow b = 5$$

$$\text{And } a + b + c = 8 \rightarrow 3 + 5 + c = 8 \rightarrow c = 0$$

Step 3 : Find the rule

The rule is therefore: $n^{\text{th}} \text{ term} = 3n^2 + 5n$

Step 4 : Check answer

For

$$n = 1, 1^{\text{st}} \text{ term} = 3(1)^2 + 5(1) = 8$$

$$n = 2, 2^{\text{nd}} \text{ term} = 3(2)^2 + 5(2) = 22$$

$$n = 3, 3^{\text{rd}} \text{ term} = 3(3)^2 + 5(3) = 42$$



Extension: Derivation of the n^{th} -term of a Quadratic Sequence

Let the n^{th} -term for a quadratic sequence be given by

$$a_n = A \cdot n^2 + B \cdot n + C \quad (20.2)$$

where A, B and C are some constants to be determined.

$$a_n = A \cdot n^2 + B \cdot n + C \quad (20.3)$$

$$a_1 = A(1)^2 + B(1) + C = A + B + C \quad (20.4)$$

$$a_2 = A(2)^2 + B(2) + C = 4A + 2B + C \quad (20.5)$$

$$a_3 = A(3)^2 + B(3) + C = 9A + 3B + C \quad (20.6)$$

$$\text{Let } d \equiv a_2 - a_1$$

$$\therefore d = 3A + B$$

$$\Rightarrow B = d - 3A \quad (20.7)$$

The common second difference is obtained from

$$\begin{aligned} D &= (a_3 - a_2) - (a_2 - a_1) \\ &= (5A + B) - (3A + B) \\ &= 2A \end{aligned}$$

$$\Rightarrow A = \frac{D}{2} \quad (20.8)$$

Therefore, from (20.7),

$$B = d - \frac{3}{2} \cdot D \quad (20.9)$$

From (20.4),

$$C = a_1 - (A + B) = a_1 - \frac{D}{2} - d + \frac{3}{2} \cdot D$$

$$\therefore C = a_1 + D - d \quad (20.10)$$

Finally, the general equation for the n^{th} -term of a quadratic sequence is given by

$$a_n = \frac{D}{2} \cdot n^2 + \left(d - \frac{3}{2}D\right) \cdot n + (a_1 - d + D) \quad (20.11)$$



Worked Example 95: Using a set of equations

Question: Study the following pattern: 1; 7; 19; 37; 61; ...

1. What is the next number in the sequence ?
2. Use variables to write an algebraic statement to generalise the pattern.
3. What will the 100th term of the sequence be ?

Answer

Step 1 : The next number in the sequence

The numbers go up in multiples of 6

$$1 + 6 = 7, \text{ then } 7 + 12 = 19$$

$$\text{Therefore } 61 + 6 \times 6 = 97$$

The next number in the sequence is 97.

Step 2 : Generalising the pattern

TERMS	1	7	19	37	61
1 st difference	6	12	18	24	
2 nd difference		6	6	6	6

The pattern will yield a quadratic equation since second difference is constant

$$\text{Therefore } an^2 + bn + c = y$$

For the first term: $n = 1$, then $y = 1$

For the second term: $n = 2$, then $y = 7$

For the third term: $n = 3$, then $y = 19$

etc....

Step 3 : Setting up sets of equations

$$a + b + c = 1 \quad (20.12)$$

$$4a + 2b + c = 7 \quad (20.13)$$

$$9a + 3b + c = 19 \quad (20.14)$$

Step 4 : Solve the sets of equations

$$\text{eqn}(2) - \text{eqn}(1) : 3a + b = 6 \quad (20.15)$$

$$\text{eqn}(3) - \text{eqn}(2) : 5a + b = 12 \quad (20.16)$$

$$\text{eqn}(5) - \text{eqn}(4) : 2a = 6 \quad (20.17)$$

$$\text{Therefore } a = 3, b = -3 \text{ and } c = 1 \quad (20.18)$$

Step 5 : Final answer

The general formula for the pattern is $3n^2 - 3n + 1$

Step 6 : Term 100

Substitute n with 100:

$$3(100)^2 - 3(100) + 1 = 29\,701$$

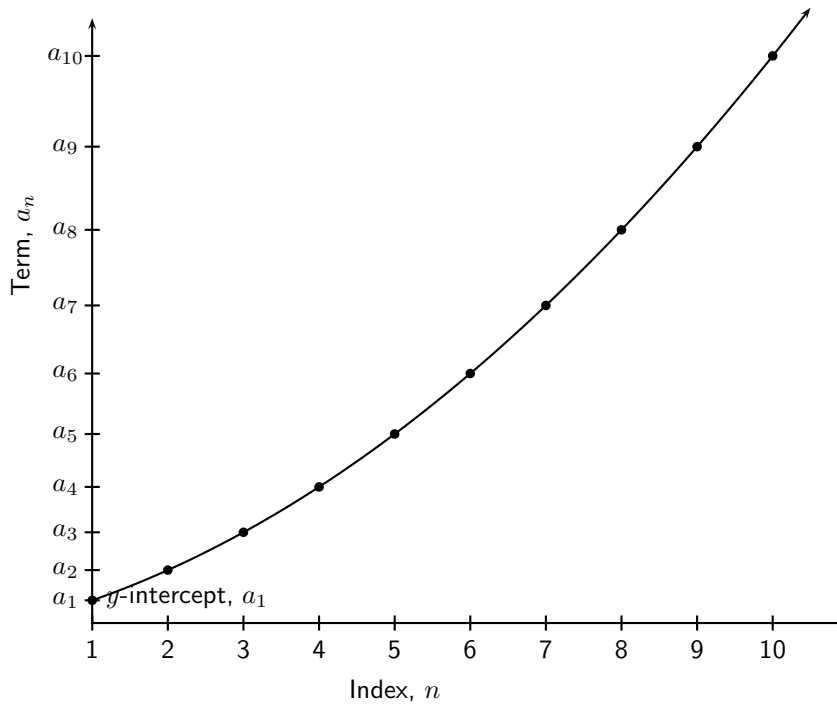
The value for term 100 is 29 701.



Extension: Plotting a graph of terms of a quadratic sequence
 Plotting a_n vs. n for a quadratic sequence yields a parabolic graph.
 Given the quadratic sequence,

$$3; 6; 10; 15; 21; \dots$$

If we plot each of the terms vs. the corresponding index, we obtain a graph of a parabola.



20.3 End of chapter Exercises

- Find the first 5 terms of the quadratic sequence defined by:

$$a_n = n^2 + 2n + 1$$

- Determine which of the following sequences is a quadratic sequence by calculating the common second difference:

- A 6, 9, 14, 21, 30, ...
- B 1, 7, 17, 31, 49, ...
- C 8, 17, 32, 53, 80, ...
- D 9, 26, 51, 84, 125, ...
- E 2, 20, 50, 92, 146, ...
- F 5, 19, 41, 71, 109, ...
- G 2, 6, 10, 14, 18, ...
- H 3, 9, 15, 21, 27, ...
- I 10, 24, 44, 70, 102, ...
- J 1, 2.5, 5, 8.5, 13, ...
- K 2.5, 6, 10.5, 16, 22.5, ...

L 0.5, 9, 20.5, 35, 52.5, ...

3. Given $a_n = 2n^2$, find for which value of n , $a_n = 242$?
4. Given $a_n = (n - 4)^2$, find for which value of n , $a_n = 36$?
5. Given $a_n = n^2 + 4$, find for which value of n , $a_n = 85$?
6. Given $a_n = 3n^2$, find a_{11} ?
7. Given $a_n = 7n^2 + 4n$, find a_9 ?
8. Given $a_n = 4n^2 + 3n - 1$, find a_5 ?
9. Given $a_n = 1,5n^2$, find a_{10} ?
10. For each of the quadratic sequences, find the common second difference, the formula for the general term and then use the formula to find a_{100} .

A 4, 7, 12, 19, 28, ...

B 2, 8, 18, 32, 50, ...

C 7, 13, 23, 37, 55, ...

D 5, 14, 29, 50, 77, ...

E 7, 22, 47, 82, 127, ...

F 3, 10, 21, 36, 55, ...

G 3, 7, 13, 21, 31, ...

H 1, 8, 27, 64, 125, ...

I 6, 13, 32, 69, 130, ...

J 2, 9, 17, 27, 39, ...

Appendix A

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