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The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Chapter 20

Quadratic Sequences - Grade 11

20.1 Introduction

In Grade 10, you learned about arithmetic sequences, where the difference between consecutive terms was constant. In this chapter we learn about quadratic sequences.

20.2 What is a quadratic sequence?

Definition: Quadratic Sequence
A quadratic sequence is a sequence of numbers in which the second differences between each consecutive term differ by the same amount, called a common second difference.

For example,

$$1;\ 2;\ 4;\ 7;\ 11;\ \ldots \tag{20.1}$$

is a quadratic sequence. Let us see why ...

If we take the difference between consecutive terms, then:

$$\begin{aligned} a_2 - a_1 &= 2 - 1 &= 1 \\ a_3 - a_2 &= 4 - 2 &= 2 \\ a_4 - a_3 &= 7 - 4 &= 3 \\ a_5 - a_4 &= 11 - 7 &= 4 \end{aligned}$$

We then work out the *second differences*, which is simply obtained by taking the difference between the consecutive differences $\{1;\ 2;\ 3;\ 4;\ \ldots\}$ obtained above:

$$\begin{aligned} 2 - 1 &= 1 \\ 3 - 2 &= 1 \\ 4 - 3 &= 1 \\ &\ldots \end{aligned}$$

We then see that the second differences are equal to 1. Thus, (20.1) is a *quadratic sequence*.

Note that the differences between consecutive terms (that is, the first differences) of a quadratic sequence form a sequence where there is a constant difference between consecutive terms. In the above example, the sequence of $\{1;\ 2;\ 3;\ 4;\ \ldots\}$, which is formed by taking the differences between consecutive terms of (20.1), has a linear formula of the kind $ax + b$.

Exercise: Quadratic Sequences

The following are also examples of quadratic sequences:

$$3;\ 6;\ 10;\ 15;\ 21;\ \ldots$$
$$4;\ 9;\ 16;\ 25;\ 36;\ \ldots$$
$$7;\ 17;\ 31;\ 49;\ 71;\ \ldots$$
$$2;\ 10;\ 26;\ 50;\ 82;\ \ldots$$
$$31;\ 30;\ 27;\ 22;\ 15;\ \ldots$$

Can you calculate the common second difference for each of the above examples?

Worked Example 93: Quadratic sequence

Question: Write down the next two terms and find a formula for the n^{th} term of the sequence $5, 12, 23, 38$,..., ...,

Answer

Step 1 : Find the first differences between the terms.

i.e. $7, 11, 15$

Step 2 : Find the 2nd differences between the terms.

the second difference is 4.

So continuing the sequence, the differences between each term will be:

$15 + 4 = 19$

$19 + 4 = 23$

Step 3 : Finding the next two terms.

So the next two terms in the sequence willl be:

$38 + 19 = 57$

$57 + 23 = 80$

So the sequence will be: $5, 12, 23, 38, 57, 80$

Step 4 : We now need to find the formula for this sequence.

We know that the first difference is 4. The start of the formula will therefore be $2n^2$.

Step 5 : We now need to work out the next part of the sequence.

If $n = 1$, you have to get the value of term 1, which is 5 in this particular sequence. The difference between $2n^2 = 2$ and original number (5) is 3, which leads to $n + 2$. Check is it works for the second term, i.e. when $n = 2$.

Then $2n^2 = 8$. The difference between term 2(12) and 8 is 4, which is can be written as $n + 2$.

So for the sequence $5, 12, 23, 38$,... the formula for the n^th term is $2n^2 + n + 2$.

General Case

If the sequence is quadratic, the n^th term should be $T_n = an^2 + bn + c$

TERMS	$a+b+c$		$4a+2b+c$		$9a+3b+c$	
1^{st} difference		$3a+b$		$5a+b$		$7a+b$
2^{nd} difference			$2a$		$2a$	

In each case, the 2nd difference is $2a$. This fact can be used to find a, then b then c.

Worked Example 94: Quadratic Sequence

Question: The following sequence is quadratic: $8, 22, 42, 68, ...$ Find the rule.

Answer

Step 1 : Assume that the rule is $an^2 + bn + c$

TERMS	8		22		42		68
1st difference		14		20		26	
2nd difference			6		6		6

Step 2 : Determine values for a, b and c

$$\begin{aligned} Then \quad & 2a = 6 \ which \ gives \ a = 3 \\ And \quad & 3a + b = 14 \rightarrow 9 + b = 14 \rightarrow b = 5 \\ And \quad & a + b + c = 8 \rightarrow 3 + 5 + c = 8 \rightarrow c = 0 \end{aligned}$$

Step 3 : Find the rule

The rule is therefore: $n^th \ term \ = 3n^2 + 5n$

Step 4 : Check answer

For

$$\begin{aligned} n &= 1, 1st \ term = 3(1)^2 + 5(1) = 8 \\ n &= 2, 2nd \ term = 3(2)^2 + 5(2) = 22 \\ n &= 3, 3rd \ term = 3(3)^2 + 5(3) = 42 \end{aligned}$$

Extension: Derivation of the n^{th}-term of a Quadratic Sequence

Let the n^{th}-term for a quadratic sequence be given by

$$a_n = A \cdot n^2 + B \cdot n + C \quad (20.2)$$

where A, B and C are some constants to be determined.

$$\begin{aligned} a_n &= A \cdot n^2 + B \cdot n + C && (20.3) \\ a_1 &= A(1)^2 + B(1) + C = A + B + C && (20.4) \\ a_2 &= A(2)^2 + B(2) + C = 4A + 2B + C && (20.5) \\ a_3 &= A(3)^2 + B(3) + C = 9A + 3B + C && (20.6) \end{aligned}$$

$$\begin{aligned} \text{Let } d &\equiv a_2 - a_1 \\ \therefore d &= 3A + B \end{aligned}$$

$$\Rightarrow B = d - 3A \quad (20.7)$$

The common second difference is obtained from

$$\begin{aligned} D &= (a_3 - a_2) - (a_2 - a_1) \\ &= (5A + B) - (3A + B) \\ &= 2A \end{aligned}$$

$$\Rightarrow A = \frac{D}{2} \quad (20.8)$$

Therefore, from (20.7),

$$B = d - \frac{3}{2} \cdot D \quad (20.9)$$

From (20.4),

$$C = a_1 - (A + B) = a_1 - \frac{D}{2} - d + \frac{3}{2} \cdot D$$

$$\therefore C = a_1 + D - d \qquad (20.10)$$

Finally, the general equation for the n^{th}-term of a quadratic sequence is given by

$$a_n = \frac{D}{2} \cdot n^2 + (d - \frac{3}{2} D) \cdot n + (a_1 - d + D) \qquad (20.11)$$

Worked Example 95: Using a set of equations

Question: Study the following pattern: 1; 7; 19; 37; 61; ...

1. What is the next number in the sequence ?
2. Use variables to write an algebraic statement to generalise the pattern.
3. What will the 100th term of the sequence be ?

Answer

Step 1 : The next number in the sequence

The numbers go up in multiples of 6
$1 + 6 = 7$, then $7 + 12 = 19$
Therefore $61 + 6 \times 6 = 97$
The next number in the sequence is 97.

Step 2 : Generalising the pattern

TERMS	1		7		19		37		61
1^{st} difference		6		12		18		24	
2^{nd} difference			6		6		6		6

The pattern will yield a quadratic equation since second difference is constant
Therefore $an^2 + bn + c = y$
For the first term: $n = 1$, then $y = 1$
For the second term: $n = 2$, then $y = 7$
For the third term: $n = 3$, then $y = 19$
etc....

Step 3 : Setting up sets of equations

$$\begin{aligned} a + b + c &= 1 \qquad (20.12) \\ 4a + 2b + c &= 7 \qquad (20.13) \\ 9a + 3b + c &= 19 \qquad (20.14) \end{aligned}$$

Step 4 : Solve the sets of equations

$$\begin{aligned} eqn(2) - eqn(1) :&\quad 3a + b = 6 \qquad (20.15) \\ eqn(3) - eqn(2) :&\quad 5a + b = 12 \qquad (20.16) \\ eqn(5) - eqn(4) :&\quad 2a = 6 \qquad (20.17) \\ Therefore &\quad a = 3, b = -3 \ and \ c = 1 \qquad (20.18) \end{aligned}$$

Step 5 : Final answer

The general formula for the pattern is $3n^2 - 3n + 1$

Step 6 : Term 100

Substitude n with 100:
$3(100)^2 - 3(100) + 1 = 29\ 701$
The value for term 100 is 29 701.

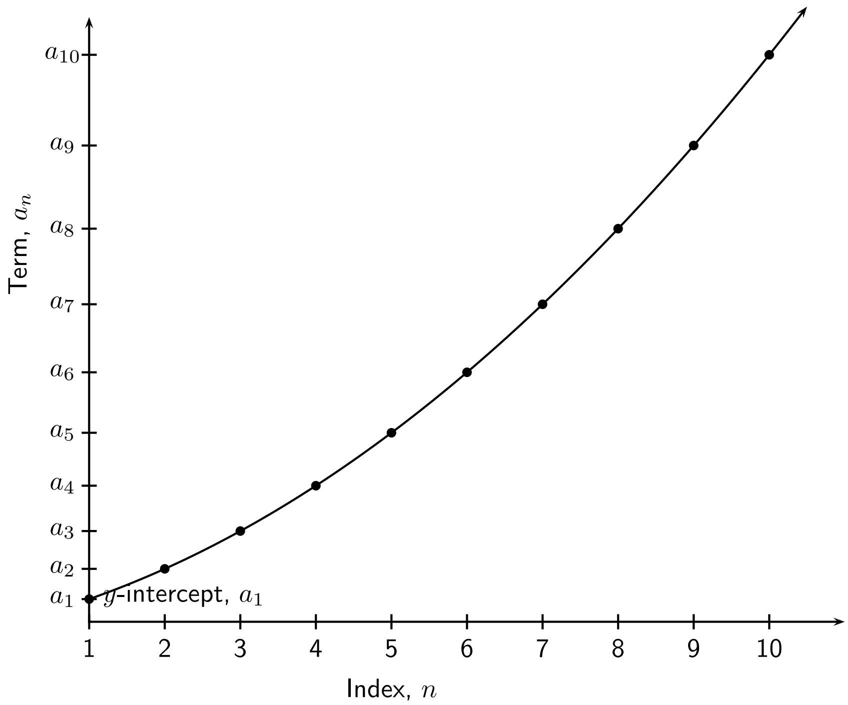
Extension: Plotting a graph of terms of a quadratic sequence

Plotting a_n vs. n for a quadratic sequence yields a parabolic graph.

Given the quadratic sequence,

$$3;\ 6;\ 10;\ 15;\ 21;\ \ldots$$

If we plot each of the terms vs. the corresponding index, we obtain a graph of a parabola.



20.3 End of chapter Exercises

1. Find the first 5 terms of the quadratic sequence defined by:

$$a_n = n^2 + 2n + 1$$

2. Determine which of the following sequences is a quadratic sequence by calculating the common second difference:

   A $6, 9, 14, 21, 30, \ldots$

   B $1, 7, 17, 31, 49, \ldots$

   C $8, 17, 32, 53, 80, \ldots$

   D $9, 26, 51, 84, 125, \ldots$

   E $2, 20, 50, 92, 146, \ldots$

   F $5, 19, 41, 71, 109, \ldots$

   G $2, 6, 10, 14, 18, \ldots$

   H $3, 9, 15, 21, 27, \ldots$

   I $10, 24, 44, 70, 102, \ldots$

   J $1, 2.5, 5, 8.5, 13, \ldots$

   K $2.5, 6, 10.5, 16, 22.5, \ldots$

L $0.5, 9, 20.5, 35, 52.5, \ldots$

3. Given $a_n = 2n^2$, find for which value of n, $a_n = 242$?
4. Given $a_n = (n-4)^2$, find for which value of n, $a_n = 36$?
5. Given $a_n = n^2 + 4$, find for which value of n, $a_n = 85$?
6. Given $a_n = 3n^2$, find a_{11}?
7. Given $a_n = 7n^2 + 4n$, find a_9?
8. Given $a_n = 4n^2 + 3n - 1$, find a_5?
9. Given $a_n = 1{,}5n^2$, find a_{10}?
10. For each of the quadratic sequences, find the common second difference, the formula for the general term and then use the formula to find a_{100}.

    A $4{,}7{,}12{,}19{,}28, \ldots$
    B $2{,}8{,}18{,}32{,}50, \ldots$
    C $7{,}13{,}23{,}37{,}55, \ldots$
    D $5{,}14{,}29{,}50{,}77, \ldots$
    E $7{,}22{,}47{,}82{,}127, \ldots$
    F $3{,}10{,}21{,}36{,}55, \ldots$
    G $3{,}7{,}13{,}21{,}31, \ldots$
    H $1{,}8{,}27{,}64{,}125, \ldots$
    I $6{,}13{,}32{,}69{,}130, \ldots$
    J $2{,}9{,}17{,}27{,}39, \ldots$

Appendix A

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