



FHSST Authors

**The Free High School Science Texts:  
Textbooks for High School Students  
Studying the Sciences  
Mathematics  
Grades 10 - 12**

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## Chapter 6

# Irrational Numbers and Rounding Off - Grade 10

### 6.1 Introduction

You have seen that repeating decimals may take a lot of paper and ink to write out. Not only is that impossible, but writing numbers out to many decimal places or a *high accuracy* is very inconvenient and rarely gives better answers. For this reason we often estimate the number to a certain number of decimal places or to a given number of *significant figures*, which is even better.

### 6.2 Irrational Numbers

---

#### Activity :: Investigation : Irrational Numbers


Which of the following cannot be written as a rational number?

**Remember:** A rational number is a fraction with numerator and denominator as integers. Terminating decimal numbers or repeating decimal numbers are rational.

1.  $\pi = 3,14159265358979323846264338327950288419716939937510\dots$
  2. 1,4
  3. 1,618 033 989 ...
  4. 100
- 

Irrational numbers are numbers that cannot be written as a rational number. You should know that a rational number can be written as a fraction with the numerator and denominator as integers. This means that any number that is *not* a terminating decimal number or a repeating decimal number are irrational. Examples of irrational numbers are:

$$\sqrt{2}, \sqrt{3}, \sqrt[3]{4}, \pi, \frac{1 + \sqrt{5}}{2} \approx 1,618\,033\,989$$

 **Important:** When irrational numbers are written in decimal form, they go on forever and there is no repeated pattern of digits.



### Important: Irrational Numbers

If you are asked to identify whether a number is rational or irrational, first write the number in decimal form. If the number is terminated then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

When you write irrational numbers in decimal form, you may (if you have a lot of time and paper!) continue writing them for many, many decimal places. However, this is not convenient and it is often necessary to round off.

## 6.3 Rounding Off

Rounding off or approximating a decimal number to a given number of decimal places is the quickest way to approximate a number. For example, if you wanted to round-off 2,6525272 to three decimal places then you would first count three places after the decimal.

$$2,652|5272$$

All numbers to the right of | are ignored after you determine whether the number in the third decimal place must be rounded up or rounded down. You *round up* the final digit if the first digit after the | was greater or equal to 5 and *round down* (leave the digit alone) otherwise.

So, since the first digit after the | is a 5, we must round up the digit in the third decimal place to a 3 and the final answer of 2,6525272 rounded to three decimal places is

$$2,653$$



### Worked Example 4: Rounding-Off

**Question:** Round-off the following numbers to the indicated number of decimal places:

1.  $\frac{120}{99} = 1,21212121\dot{2}$  to 3 decimal places
2.  $\pi = 3,141592654\dots$  to 4 decimal places
3.  $\sqrt{3} = 1,7320508\dots$  to 4 decimal places

**Answer**

**Step 1 :** Determine the last digit that is kept and mark the cut-off point with |.

1.  $\frac{120}{99} = 1,212|12121\dot{2}$
2.  $\pi = 3,1415|92654\dots$
3.  $\sqrt{3} = 1,7320|508\dots$

**Step 2 :** Determine whether the last digit is rounded up or down.

1. The last digit of  $\frac{120}{99} = 1,212|12121\dot{2}$  must be rounded-down.
2. The last digit of  $\pi = 3,1415|92654\dots$  must be rounded-up.
3. The last digit of  $\sqrt{3} = 1,7320|508\dots$  must be rounded-up.

**Step 3 :** Write the final answer.

1.  $\frac{120}{99} = 1,212$  rounded to 3 decimal places
2.  $\pi = 3,1416$  rounded to 4 decimal places
3.  $\sqrt{3} = 1,7321$  rounded to 4 decimal places

## 6.4 End of Chapter Exercises

- Write the following rational numbers to 2 decimal places:
  - $\frac{1}{2}$
  - 1
  - $0,11111\bar{1}$
  - $0,99999\bar{1}$
- Write the following irrational numbers to 2 decimal places:
  - 3,141592654...
  - 1,618 033 989 ...
  - 1,41421356 ...
  - 2,71828182845904523536 ...
- Use your calculator and write the following irrational numbers to 3 decimal places:
  - $\sqrt{2}$
  - $\sqrt{3}$
  - $\sqrt{5}$
  - $\sqrt{6}$
- Use your calculator (where necessary) and write the following irrational numbers to 5 decimal places:
  - $\sqrt{8}$
  - $\sqrt{768}$
  - $\sqrt{100}$
  - $\sqrt{0,49}$
  - $\sqrt{0,0016}$
  - $\sqrt{0,25}$
  - $\sqrt{36}$
  - $\sqrt{1960}$
  - $\sqrt{0,0036}$
  - $-8\sqrt{0,04}$
  - $5\sqrt{80}$
- Write the following irrational numbers to 3 decimal places and then write them as a rational number to get an approximation to the irrational number. For example,  $\sqrt{3} = 1,73205\dots$ . To 3 decimal places,  $\sqrt{3} = 1,732$ .  $1,732 = 1\frac{732}{1000} = 1\frac{183}{250}$ . Therefore,  $\sqrt{3}$  is approximately  $1\frac{183}{250}$ .
  - 3,141592654...
  - 1,618 033 989 ...
  - 1,41421356 ...
  - 2,71828182845904523536 ...





## Appendix A

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